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Chatziioannou, V., & Van Walstijn, M. (2013). An Energy Conserving Finite Difference Scheme for Simulation of Collisions. 584-591. Paper presented at SMC Sound and Music Computing Conference, Stockholm, Sweden.

Document Version:

Publisher's PDF, also known as Version of record

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AN ENERGY CONSERVING FINITE DIFFERENCE SCHEME FOR SIMULATION OF COLLISIONS

Vasileios Chatziioannou

Institute of Music Acoustics
University of Music and Performing Arts Vienna
chatziioannou@mdw.ac.at

Maarten van Walstijn

Sonic Arts Research Centre
Queen's University Belfast
m.vanwalstijn@qub.ac.uk

ABSTRACT

Nonlinear phenomena play an essential role in the sound production process of many musical instruments. A common source of these effects is object collision, the numerical simulation of which is known to give rise to stability issues. This paper presents a method to construct numerical schemes that conserve the total energy in simulations of one-mass systems involving collisions, with no conditions imposed on any of the physical or numerical parameters. This facilitates the adaptation of numerical models to experimental data, and allows a more free parameter adjustment in sound synthesis explorations. The energy preservedness of the proposed method is tested and demonstrated through several examples, including a bouncing ball and a non-linear oscillator, and implications regarding the wider applicability are discussed.

1. INTRODUCTION

Impact modelling is required in many engineering problems, for example during the simulation of colliding or bouncing objects [1]. Taking Hertz's contact law as a starting point [2] and denoting the compression along the displacement axis y with Δy , collision forces can generally be modelled using a one-sided power law

$$f(\Delta y) = \begin{cases} k_c \Delta y^\alpha & \text{if } \Delta y > 0 \\ 0 & \text{if } \Delta y \leq 0 \end{cases}, \quad (1)$$

where the force f is active only for positive compression values, and where k_c and α are power law constants.

In the context of musical acoustics, collisions have often been studied in relation to hammer and mallet impacts. For example, experimental studies of hammer-string interaction in a piano have reported exponent values in the range of $\alpha \in [2, 5]$ [3], though, in principle, α may take on any value larger than 1 for impact modelling [4]. Collisions may also occur in a more spatially distributed manner, such as the string-bridge interaction in a sitar. In all cases, the impactive interaction represents an important nonlinear element in the system that is closely linked to the expressive control and characteristics of the instrument.

The musical acoustics and sound computing literature offers a variety of time-stepping methods for simulating collisions, most of which are based on finite differences (e.g. [2, 5]) or closely related methods such as the trapezoidal rule [6] or Verlet integration (e.g. [1, 7]). While many successful simulation results have been obtained, and stability can even be shown for some specific cases or under specific assumptions (see, e.g. [8]), the formulation of a more general class of provably stable algorithms for impact modelling is still considered as an open and difficult problem [1, 5]. This sets collision modelling problems somewhat apart from most other challenges naturally appearing in simulation of musical instruments. That is, the past decade has seen a significant development of energy methods in finite difference simulation of musical instruments and parts thereof, notably in [5] and further publications by the same author. As such, provably stable schemes have been derived for a wide range of systems, including nonlinearly vibrating drums [9] and shells [10]. The general approach taken herein is that difference operators are applied to the Newtonian description of the system, the stability bounds of which are established through defining an invariant representing the numerical counterpart of the Hamiltonian of the underlying system. However this way of deriving schemes has limitations in application to systems in which the force is a non-smooth function of the phase space variables, in which case the invariant can only be defined for specific model parameters [5].

The present authors propose to address this by first reformulating the system in its Hamiltonian form [11], and discretise this rather than Newton's equations of motion. Drawing from a wider research field, it can be said that Hamilton's equations can generally be discretised using two different approaches [12]. The first approach leads to numerical schemes that preserve the symplectic structure of the system and allow only canonical transformations in each integration step, while the second approach aims to preserve the Hamiltonian of the system; it has been shown that only one of these properties can generally be preserved [13]. A fundamental observation is that symplectic schemes impose a stronger constraint on the behaviour of the numerical solution while preserving a slightly perturbed Hamiltonian. Since symplecticity is more suited to the study of families of trajectories and long-term behaviour of dynamical systems, this approach has dominated much of the physics and engineering oriented research. There are however indications in the literature that

energy-conserving schemes possess better stability properties than symplectic methods (see, e.g. [14]). This is particularly relevant for real-time sound synthesis applications, in which stability has to be guaranteed with minimal constraints on any of the model parameters. An energy-conserving approach is therefore adopted in this paper, focusing on a small set of simplified test problems involving a point mass colliding with a rigid barrier.

2. NUMERICAL MODELLING OF IMPACTS

The most basic model employing (1) is that of a point-mass colliding with a rigid barrier positioned at $y = 0$, where the mass approaches the barrier from below ($y < 0$). The motion of the mass is then governed by

$$m \frac{d^2 y}{dt^2} + k_c [y]^\alpha = 0, \quad (2)$$

where m is the object mass and

$$[y] = \begin{cases} y & \text{if } y > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Since we are aiming at the construction of energy preserving schemes, no dissipative components are included at this point, but as explained in Section 3, these can be added once the stability properties have been established. It has been shown in [5] that while simply applying a centered difference operator to the acceleration term in (2) leads to an unstable scheme, partially conservative behaviour can be ensured for the specific cases $\alpha = 1$ and $\alpha = 3$ with the use of an average operator. For instance if y_n denotes the value of variable y at time $n\Delta t$, with Δt being the sampling interval, then the following numerical scheme for a cubic power law

$$m \frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta t^2} + k_c [y_n]^2 \frac{y_{n+1} + y_{n-1}}{2} = 0 \quad (4)$$

preserves the energy-like function

$$H_n = \frac{1}{2} m \left(\frac{y_n - y_{n-1}}{\Delta t} \right)^2 + \frac{1}{4} k_c [y_n]^2 [y_{n-1}]^2 \quad (5)$$

in the two main phases of the simulation ($y \leq 0$ and $y > 0$). The main downside of directly discretising the Newtonian equation of motion (2) is that nothing firm can be stated about stability of simulations with values of α other than 1 or 3, since an expression analogous to (5) is then not forthcoming [5].

2.1 Hamiltonian formulation

Aiming at a more general treatment of power-law nonlinearities, we attempt to construct an energy preserving scheme for an impact force of type (1) with arbitrary exponent $\alpha \geq 1$, starting from Hamilton's equations. The equivalent Hamiltonian formulation of (2) is

$$\frac{dy}{dt} = \frac{\partial H}{\partial p}, \quad (6a)$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial y}, \quad (6b)$$

where

$$H(y, p) = \frac{p^2}{2m} + \frac{k_c}{\alpha + 1} [y]^{\alpha+1}, \quad (7)$$

is the Hamiltonian of the system and p is the momentum of the mass. Employing mid-point derivative approximations, system (6) can be discretised to yield the numerical scheme:

$$\frac{y_{n+1} - y_n}{\Delta t} = \frac{1}{2m} \frac{p_{n+1}^2 - p_n^2}{p_{n+1} - p_n}, \quad (8a)$$

$$\frac{p_{n+1} - p_n}{\Delta t} = -\frac{k_c}{\alpha + 1} \frac{[y_{n+1}]^{\alpha+1} - [y_n]^{\alpha+1}}{y_{n+1} - y_n}. \quad (8b)$$

Now setting

$$\begin{cases} q_n &= p_n \Delta t / m \\ \beta &= \Delta t^2 k_c / m \end{cases}, \quad (9)$$

yields a scheme with just two parameters:

$$y_{n+1} - y_n = \frac{1}{2} (q_{n+1} + q_n), \quad (10a)$$

$$q_{n+1} - q_n = -\frac{\beta}{\alpha + 1} \frac{[y_{n+1}]^{\alpha+1} - [y_n]^{\alpha+1}}{y_{n+1} - y_n}. \quad (10b)$$

Solving (10) is facilitated by defining the auxiliary variable

$$x = \frac{1}{2} (q_{n+1} + q_n), \quad (11)$$

which, from equation (10a), gives

$$\begin{aligned} q_{n+1} &= 2x - q_n, \\ y_{n+1} &= y_n + x. \end{aligned} \quad (12)$$

Substituting into equation (10b) we have:

$$\begin{aligned} \frac{\beta}{2(\alpha + 1)} \left(\frac{[y_n + x]^{\alpha+1} - [y_n]^{\alpha+1}}{x} \right) + x - q_n &= 0 \\ \Rightarrow F(x) &= 0. \end{aligned} \quad (13)$$

Note that

$$\lim_{x \rightarrow 0} F(x) = \frac{\beta}{2} [y_n]^\alpha - q_n, \quad (14)$$

so there is no singularity in $F(x)$. To sum up, the Hamiltonian system is discretised in (8) and subsequently transformed in (10), whereas for the computation (13) is solved numerically to yield a physically correct root of $F(x)$ (see Section 2.1.2), which is used to update y and q using (12).

2.1.1 Conservation of Energy

The presented scheme can be shown to conserve the total system energy at each time step as follows. Rewriting (8) as

$$\frac{1}{\Delta t} (y_{n+1} - y_n) (p_{n+1} - p_n) = \frac{1}{2m} (p_{n+1}^2 - p_n^2) \quad (15a)$$

$$\begin{aligned} \frac{1}{\Delta t} (y_{n+1} - y_n) (p_{n+1} - p_n) &= \\ -\frac{k_c}{\alpha + 1} ([y_{n+1}]^{\alpha+1} - [y_n]^{\alpha+1}) \end{aligned} \quad (15b)$$

and substituting by parts yields

$$\left(\frac{p_{n+1}^2}{2m} + \frac{k_c}{\alpha+1} [y_{n+1}]^{\alpha+1} \right) = \left(\frac{p_n^2}{2m} + \frac{k_c}{\alpha+1} [y_n]^{\alpha+1} \right) \\ \Rightarrow H(y_{n+1}, p_{n+1}) = H(y_n, p_n). \quad (16)$$

2.1.2 Existence and Uniqueness

The scheme relies on finding a solution to equation (13), which can be achieved numerically provided that a solution exists. From the definition of $F(x)$ it follows that

$$\frac{dF}{dx} = 1 + A \frac{(\alpha+1)[y_n+x]^\alpha x - [y_n+x]^{\alpha+1} + [y_n]^{\alpha+1}}{x^2}, \quad (17)$$

with $A = \frac{\beta}{2(\alpha+1)}$ and $\lim_{x \rightarrow 0} \frac{dF}{dx} = 1 + \frac{\alpha\beta}{4} [y_n]^{\alpha-1}$.

It can be shown that $dF/dx \geq 1$, meaning that $F(x)$ always has a single root. This is equivalent to showing that

$$G(y_n + x) \leq G(y_n) + x G'(y_n + x), \quad (18)$$

where

$$G(y) = [y]^{\alpha+1} \\ G'(y) = dG/dy = (\alpha+1)[y]^\alpha. \quad (19)$$

Given that $G(y)$ is a convex function, the inequality (18) holds $\forall y_n \in \mathbb{R}$, and this result is independent of the value of q_n . Hence under the condition $\alpha \geq 1$, a unique solution of (13) exists, regardless of the value of β . Since $F(x)$ is near-linear in the neighbourhood of its root, the solution can be found with excellent convergence using the Newton-Raphson method; the number of iterations required can be kept low (typically below 6) by using the previous value of x as the initial guess.

2.2 Energy preservedness under finite precision

Due to quantisation in finite-precision arithmetic, the Hamiltonian can be preserved only to machine precision in implementations on digital processors. The resulting energy error can be expressed in terms of the deviation of $H_n = H(y_n, p_n)$ from the initial energy H_0 , which in normalised form reads

$$e_n = \left| \frac{H_n - H_0}{H_0} \right|. \quad (20)$$

It is worth noting that quantisation generally results into a random-like signal e_n that is zero mean and as such will not cause an energy shift over time. Figures 1 and 2 show examples of the mass trajectory and the associated e_n obtained with the proposed scheme (8) (labeled FD_H for discretising Hamilton's equations using finite differences). For comparison, the corresponding results for $\alpha = 1$ and $\alpha = 3$ calculated with the partially stable finite difference schemes presented in [5] are also shown (labeled FD_N for discretising Newton's second law).

In order to get a more complete view of the energy preservation properties of the proposed scheme, its performance is analysed across a range of α and β values. The variations in these parameters correspond to different levels of

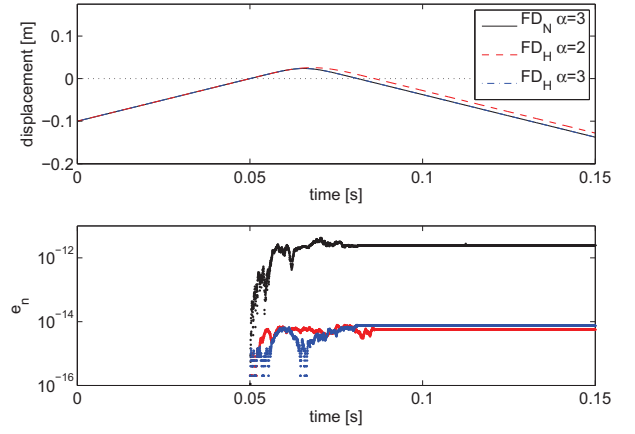


Figure 1. Simulation of a unit mass ($m = 1$ kg) colliding with a rigid barrier with initial position $y_0 = -0.1$ m and momentum $p_0 = 2$ kg m/s. The stiffness is chosen as $k_c = \sqrt{5000}^{\alpha+1}$. Top: mass displacement. Bottom: energy error by (20). All simulations were run at a 44.1 kHz sampling rate.

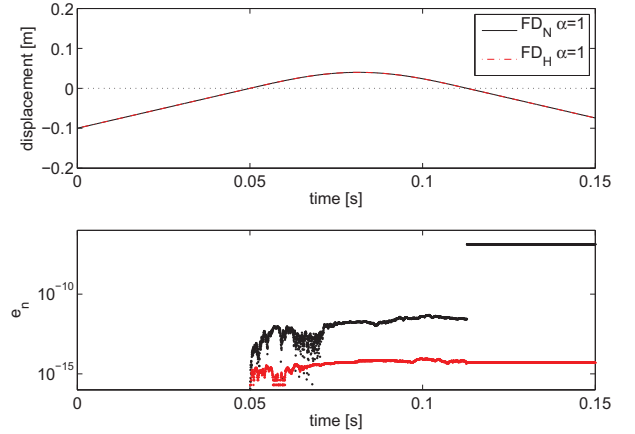


Figure 2. Simulation of a unit point-mass approaching a rigid barrier with initial position $y_0 = -0.1$ m and momentum $p_0 = 2$ kg m/s with $k_c = 2.5e3$ and a sampling rate of 44.1 kHz Top: mass displacement. Bottom: energy error.

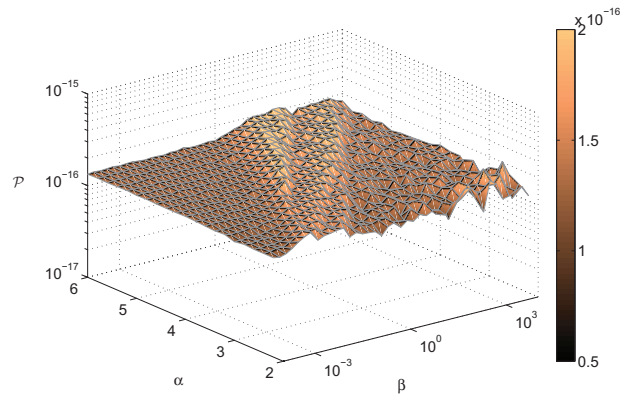


Figure 3. Simulation results of the energy preservation metric (21) as a function of α and β .

interaction between the mass and the barrier. To ensure a meaningful comparison, the calculations are made independent of the collision duration and the initial energy of the system, using the following energy preservation metric:

$$\mathcal{P} = \sum_{n=n_1}^{n_2} \frac{|H_{n+1} - H_n|}{(n_2 - n_1 + 1)H_0}, \quad (21)$$

where the collision occurs in the interval $[n_1, n_2]$. \mathcal{P} can be thought of as the mean energy deviation per sample during the contact period, thus excluding periods during which energy deviations are expected to be negligible. As depicted in Figure 3 the preservedness is only mildly dependent on the model parameters, and structurally retains very low values. This result supports a strong confidence in the stability of practical implementations.

2.3 Effective repelling force

Having established the stability properties, the immediate next question to explore is how well the scheme approximates the original continuous-time model. While standard finite difference procedures may be used to show that the scheme is of second order accuracy, additional insight can be obtained by determining the extent to which Newton's second law $f = m \partial^2 y / \partial t^2 = \partial p / \partial t$ is adhered to. This can be done by defining the *effective repelling force* of the scheme as

$$\begin{aligned} f_{n+\frac{1}{2}} &= \frac{p_{n+1} - p_n}{\Delta t} \\ &= \frac{k_c}{\alpha + 1} \left(\frac{|y_n + x|^{\alpha+1} - |y_n|^{\alpha+1}}{x} \right), \end{aligned} \quad (22)$$

where we made use of (8b) and (12). Note that $x = (q_{n+1} + q_n)/2$ can be thought of as the mid-point value $q_{n+\frac{1}{2}}$, thus representing a measure of momentum. In other words, the accuracy of equation (22) in approximating the underlying power-law depends directly on the impact momentum, and the scheme converges to (1) in the limit:

$$\lim_{x \rightarrow 0} f_{n+\frac{1}{2}} = k_c |y|^\alpha. \quad (23)$$

Given that $x \rightarrow 0$ when $\Delta t \rightarrow 0$, this also demonstrates that the numerical model is consistent with theory.

Figure 4 shows two examples of plotting the absolute value of effective repelling force, as directly evaluated from simulation data, against the mid-point displacement $(y_{n+1} + y_n)/2$, and comparing to the corresponding theoretical term $k_c [(y_{n+1} + y_n)/2]^\alpha$. For visual clarity, the values for β and q_0 have deliberately been chosen high; the discrepancy between the effective repelling force and its theoretical counterpart is considerably smaller for lower values. The more important notion that can be derived from these plots is that the scheme effectively smoothes the curve around $y = 0$, leading to a continuously differentiable force function, which can be shown to be of class C^α .

2.4 Generalisation

The conservation of energy can be shown to hold for a more general class of nonlinear one-mass oscillators, represented by a generic Hamiltonian. For an arbitrary $H(y, p)$,

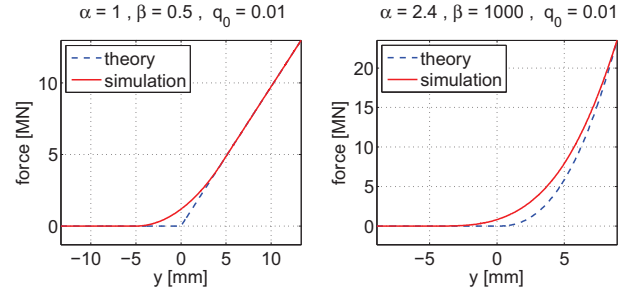


Figure 4. Examples of the effective repelling force for a power law of linear form (left) and cubic form (right). In both cases, a unit mass and a sampling rate of 44.1 kHz was used to evaluate the force curves. The model parameters used are shown above the plots.

applying the following mid-point derivative approximations

$$\frac{y_{n+1} - y_n}{\Delta t} = \frac{H(y_n, p_{n+1}) - H(y_n, p_n)}{p_{n+1} - p_n} \quad (24a)$$

$$\frac{p_{n+1} - p_n}{\Delta t} = -\frac{H(y_{n+1}, p_{n+1}) - H(y_n, p_{n+1})}{y_{n+1} - y_n}, \quad (24b)$$

yields a general numerical scheme for which, as previously, energy conservation follows from

$$\begin{aligned} \frac{1}{\Delta t} (y_{n+1} - y_n) (p_{n+1} - p_n) &= H(y_n, p_{n+1}) - H(y_n, p_n) \\ \frac{1}{\Delta t} (y_{n+1} - y_n) (p_{n+1} - p_n) &= \\ &= -(H(y_{n+1}, p_{n+1}) - H(y_n, p_{n+1})), \end{aligned}$$

hence

$$H(y_{n+1}, p_{n+1}) = H(y_n, p_n). \quad (25)$$

A beneficial feature of the method is that - unlike equation (5) - the total energy at each time step n is calculated from the state space variables in exactly the same way as for the continuous system, and is evaluated using the values of a single time step. In other words, the operator H renders the energy invariant in both domains.

3. FURTHER EXAMPLES

With no specific constraints on the Hamiltonian, provably stable algorithms can be derived for a wider class of one-mass systems involving collisions. In order to demonstrate this, three further cases are discussed here, simulated using a sampling rate of 44.1 kHz. For each case, it can be shown that the nonlinear equation analogous to equation (13) always has a unique solution, but the proofs are omitted here for brevity.

3.1 Bouncing Ball

Consider a ball falling under gravity and bouncing on a floor (at $y = 0$), neglecting any frictional effects. The Hamiltonian of the system is [15]

$$H = \frac{p^2}{2m} + \frac{k_c}{\alpha + 1} [-y]^{\alpha+1} + m g_0 y, \quad (26)$$

where g_0 is the gravitational acceleration. The Hamiltonian formulation (6) is discretised in the same way as explained in section 2.1, yielding the nonlinear function

$$F(x) = \frac{\beta}{2(\alpha+1)} \frac{[-y_n - x]^{\alpha+1} - [-y_n]^{\alpha+1}}{x} \quad (27)$$

$$+ x - q_n + \Delta t^2 g/2 = 0,$$

where for $x \rightarrow 0$ the first term is defined in a way similar to (14). Figure 5 shows the results of such a simulation with $\alpha = 4$; due to the lack of losses the ball bounces back to its initial height and the energy is conserved.

3.2 Oscillating mass with repelling force

So far, only a mass colliding with a barrier has been considered. The system begins to bear a little more resemblance to a musical instrument if the oscillating element can store potential energy in a spring of stiffness k . The repelling force is now set to become active above a specified displacement y_0 . The Hamiltonian of this system is

$$H = \frac{p^2}{2m} + \frac{k}{2}y^2 + \frac{k_c}{\alpha+1}[y - y_0]^{\alpha+1}. \quad (28)$$

The corresponding nonlinear function is now

$$F(x) = \frac{\beta}{2(\alpha+1)} \frac{[y_n - y_0 + x]^{\alpha+1} - [y_n - y_0]^{\alpha+1}}{x} \quad (29)$$

$$+ x - q_n + \frac{\Delta t^2 k}{4m}(x + 2y_n) = 0.$$

Figure 6 shows the result of an example simulation using $\alpha = 2$. As can be seen, the repetitive collisions do not cause an accumulative energy shift, and the energy is conserved to machine precision. This was observed for a large number of simulations with different parameters and long simulation times.

3.3 Non-conservative systems

In more realistic scenarios, the total energy of the system is not conserved. This can occur due to damping effects or the application of non-conservative external forces. For instance, the Newtonian equation of motion

$$m \frac{d^2 y}{dt^2} + m\gamma \frac{dy}{dt} - k_c [-y]^\alpha + m g_0 = f, \quad (30)$$

describes the displacement of a bouncing ball subject to an external force f as well as a resistive term that represents frictional losses, where γ is a damping constant. The corresponding Hamiltonian, which is now time-dependent, can be found using the so-called Caldirola–Kanai Lagrangian [16, 17]:

$$H = e^{-\gamma t} \frac{p^2}{2m} + e^{\gamma t} \left(\frac{k_c}{\alpha+1} [-y]^{\alpha+1} + m g_0 y \right) \quad (31)$$

and Hamilton's equations for this system, including the application of the external force, are

$$\frac{dy}{dt} = \frac{\partial H}{\partial p}, \quad (32a)$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial y} + e^{\gamma t} f, \quad (32b)$$

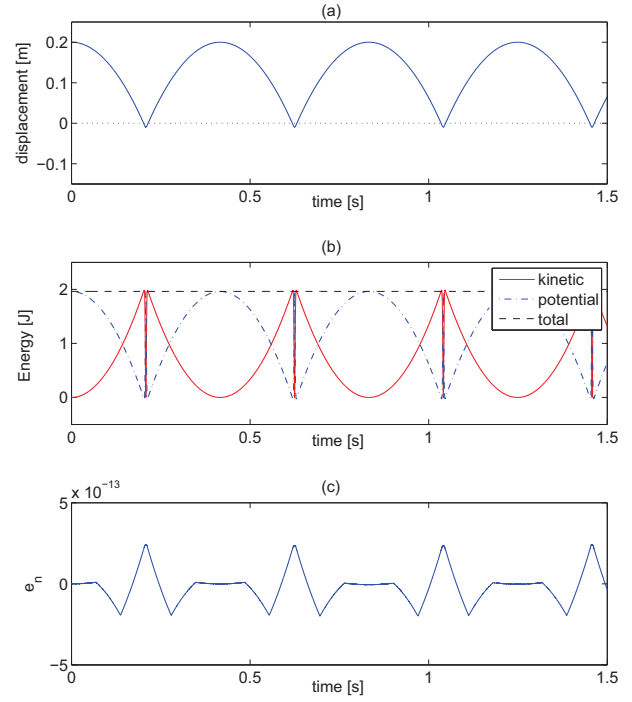


Figure 5. (a) Displacement of a lossless bouncing ball under a gravitational force with $k_c = 1e11$ and $\alpha = 4$. (b) The energy components. (c) The energy error.

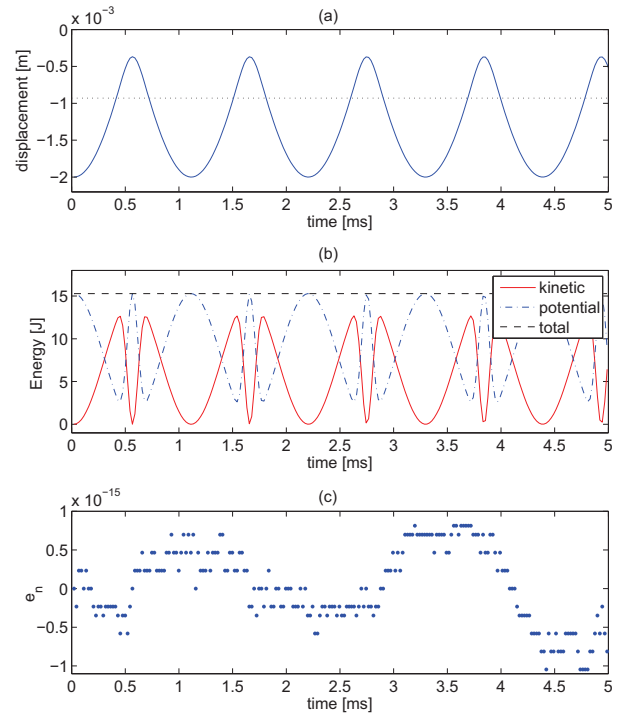


Figure 6. (a) Displacement of a lossless oscillating unit mass attached to a spring with stiffness $k = (2\pi 440)^2$ N/m and initial position $y_0 = -2$ mm. A repelling force becomes active when $y > -0.93$ mm following a quadratic power law with $k_c = 2.5e10$. (b) The corresponding energy components. (c) The energy error.

Note that in this case p represents the generalised momentum and equals $\partial L / \partial \dot{y} = e^{\gamma t} m \dot{y}$, where L is the Lagrangian and $\dot{y} = dy/dt$. Hence the total (internal) energy of the system is given by $e^{-\gamma t} H$. The partial derivatives of the Hamiltonian are now defined at mid-point as:

$$\frac{\partial H}{\partial p} \Big|_{t=t+\Delta t/2} \approx e^{-\gamma(n+\frac{1}{2})\Delta t} \frac{1}{m} \frac{p_{n+1} + p_n}{2}, \quad (33a)$$

$$\frac{\partial H}{\partial y} \Big|_{t=t+\Delta t/2} \approx e^{\gamma(n+\frac{1}{2})\Delta t} \frac{k_c}{\alpha + 1} \frac{[y_{n+1}]^{\alpha+1} - [y_n]^{\alpha+1}}{y_{n+1} - y_n}, \quad (33b)$$

and mid-point evaluation of the external force term yields

$$e^{\gamma t} f \Big|_{t=t+\Delta t/2} \approx e^{\gamma(n+\frac{1}{2})\Delta t} \frac{f_{n+1} + f_n}{2}. \quad (34)$$

Applying these to (32) and defining

$$\left\{ \begin{array}{l} q_n = \frac{\Delta t}{m} e^{-\gamma n \Delta t} p_n \\ w_n = \frac{\Delta t^2}{m} f_n \end{array} \right\}, \quad (35)$$

allows to write the resulting scheme as

$$\begin{aligned} y_{n+1} - y_n &= \frac{r q_{n+1} + r^{-1} q_n}{2}, \\ r q_{n+1} - r^{-1} q_n &= \frac{w_{n+1} + w_n}{2} \\ &\quad - \frac{\beta}{2(\alpha + 1)} \frac{[y_{n+1}]^{\alpha+1} - [y_n]^{\alpha+1}}{y_{n+1} - y_n}, \end{aligned} \quad (36a) \quad (36b)$$

where $r = e^{\gamma \Delta t/2}$ and β is defined again as in (13). Solution is now facilitated by defining the auxiliary variable as

$$x = \frac{1}{2}(r q_{n+1} + r^{-1} q_n), \quad (37)$$

which again yields a nonlinear equation to be solved:

$$\begin{aligned} F(x) &= \frac{\beta}{2(\alpha + 1)} \frac{[-y_n - x]^{\alpha+1} - [-y_n]^{\alpha+1}}{x} \\ &\quad + x - q_n/r - \frac{w_{n+1} + w_n}{4} = 0. \end{aligned} \quad (38)$$

Figure 7 shows the simulation results for a mass, initially driven by an external force, with its motion being damped by frictional forces. An energy preservation check does not apply now, but stability may still be observed in that

$$\frac{\partial (e^{-\gamma t} H)}{\partial t} \leq 0 \quad (39)$$

for any period during which $f = 0$. Regarding finite precision effects, checking the simulation after 20 seconds run time verified that the oscillations decay to zero, i.e. no limit cycles appear.

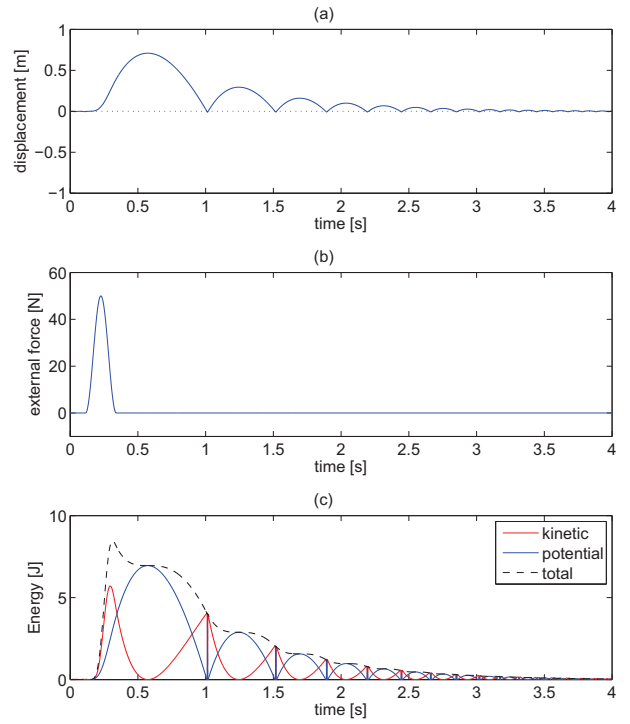


Figure 7. (a) Displacement of a unit mass driven by an external force, with initial conditions $y_0, p_0 = 0$ and internal damping $\gamma = 2$. (b) The profile of the external force. (c) The variation of the energy components.

4. CONCLUSIONS AND PERSPECTIVES

A method has been presented to formulate energy preserving schemes for the simulation of a point mass under the influence of a nonlinear force term. This has been achieved by discretising the Hamiltonian formulation instead of the Newtonian equation of motion. A proof of existence and uniqueness of the solution has been given for the case of an impactive interaction governed by a power law. The accuracy of the scheme has been investigated through the effective repelling force, which is dependent on the power-law constants and the impact velocity. Simulation results with several lossless example systems have confirmed that the system energy is conserved to machine precision, regardless of the model parameters.

For lossless one-mass systems, the proposed method is similar to that presented by Greenspan [18]. That is, for problems of the form

$$m \frac{d^2 y}{dt^2} = f(y), \quad (40)$$

where the force f is a nonlinear function of y , scheme (24) is equivalent to Greenspan's method, which uses the potential function rather than the Hamiltonian as its starting point. A common feature between the proposed method and [18] is that finite difference operators are directly applied to an energy variable, as distinct from arriving at a scheme by applying difference operators to the variables of a Newtonian description, which has been the prevalent approach to derive numerical schemes for musical acoustics and sound computing applications. It is worth noting

that for many musically relevant systems, including any of its linear components, such direct discretisation of energy variables holds no particular advantages; the specific merit of applying difference operators to the Hamiltonian itself only emerges in application to systems in which the force is a non-smooth function of the phase space variables, making it particularly suitable for simulation of collisions. In comparison to [18], the advantage of approximating partial derivatives of the Hamiltonian rather than the potential function is that it allows direct extension to problems of the form

$$m \frac{d^2 y}{dt^2} = f(y, p), \quad (41)$$

such as collision models with nonlinear damping [19]. Given that all dynamic systems can be formulated in Hamiltonian form, one could go one step further here and conjecture that the approach can be applied to more complex systems, which would open up new possibilities for the simulation of musical instrument sounds. This would invariably involve impacting of spatially distributed elements (e.g. strings, membranes, plates). In order to gain some initial perspective of how the proposed method would apply to such systems, consider a problem of the form of (40) where $f(y) = ky$ is a simple linear spring restoring force. The generalised scheme (24) then reduces to

$$\frac{y_{n+1} - y_n}{\Delta t} = \frac{1}{m} \left(\frac{p_{n+1} + p_n}{2} \right) \quad (42a)$$

$$\frac{p_{n+1} - p_n}{\Delta t} = -k \left(\frac{y_{n+1} + y_n}{2} \right), \quad (42b)$$

which is equivalent to applying the trapezoidal integration rule to $\partial y / \partial t = p/m$, $\partial p / \partial t = -ky$, thus shifting the system resonance frequency $\sqrt{k/m}$ in the same way as the bilinear transform. This signifies that the proposed manner of discretisation results into rather heavy numerical dispersion for any linear subsystem. While it is straightforward to pre-compensate for such errors in the case of a one-mass system, the implications for a mass interacting with a spatially distributed object are more complex, and worthy of further investigation. Other key questions to be addressed in future research are whether any uniqueness issues would arise and how these may be resolved, and how the resulting conservative schemes compare to alternatives, in particular symplectic schemes.

5. REFERENCES

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